

Density of ϕ -Polynomials

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Let ϕ be a real function continuous on the real line. Let $V_n(\phi)$ be the family of functions of the form

$$F(A, x) = \sum_{k=1}^n a_k \phi(a_{n+k}x), \quad a_1, \dots, a_{2n} \text{ real.}$$

The problem of approximation by $V_n(\phi)$ is a special case of that of approximation by a curve of functions or equivalently approximation by γ -polynomials [1–3]. A function in $V_n(\phi)$, for some n , is called a ϕ -polynomial. We are interested in determining sets of functions in which the set of ϕ -polynomials is dense, with respect to the uniform (Chebyshev) norm.

DEFINITION. A subset \mathcal{G} of a set of functions \mathcal{F} is fundamental in \mathcal{F} if the set of (finite) linear combinations of functions in \mathcal{G} is dense in \mathcal{F} .

THEOREM. Let \mathcal{F} be a set of functions defined on a finite interval $[\alpha, \beta]$ of the real line. Let ϕ have a Taylor series $\sum_{k=0}^{\infty} c_k x^k$, absolutely convergent in a closed neighborhood I of zero, such that the set of powers x^k with nonzero coefficient c_k is fundamental in \mathcal{F} . If $\{\gamma_k\}$ is an infinite sequence of nonzero numbers converging to zero, then the set $\{\phi(0), \phi(\gamma_1 x), \phi(\gamma_2 x), \dots\}$ is fundamental in \mathcal{F} .

Proof. It suffices to show that each power x^n with a nonzero coefficient c_n can be uniformly approximated by linear combinations of $\{\phi(0), \phi(\gamma_1 x), \phi(\gamma_2 x), \dots\}$. We shall prove that for each $j > 0$ there is a sequence of linear combinations with the k th linear combination equal to $x^n + O(k^{-j})$ on $[\alpha, \beta]$. There exists a positive constant $\mu \leq 1$ such that $\mu x \in I$ for $x \in [\alpha, \beta]$. Assume without loss of generality that $|\gamma_i| \leq \mu$ for all i , then $\phi(\gamma_i x)$ has a Taylor series convergent on $[\alpha, \beta]$ for all i . Suppose $c_1 \neq 0$. Select γ_i with $|\gamma_i| \leq 1/k^j$ and $|\gamma_i| < \mu^3$.

$$\frac{1}{\gamma_i c_1} [\phi(\gamma_i x) - \phi(0)] = x + \frac{c_2}{c_1} x^2 \gamma_i + \frac{c_3}{c_1} x^3 \gamma_i^2 + \cdots \\ + x + \gamma_i \left(\frac{c_2}{c_1} x^2 \right) + \frac{\gamma_i}{c_1} \sum_{k=3}^j c_k x^k \gamma_i^{k-2}.$$

Now for $x \in [\alpha, \beta]$,

$$\sum_{k=3}^{\infty} |c_k x^k \gamma_i^{k-2}| \leq \sum_{k=3}^{\infty} |c_k x^k| |\gamma_i|^{k/3} \leq \sum_{k=3}^{\infty} |c_k| |\mu x|^k < \infty.$$

Now let us suppose that our assertion is true for all powers up to and including the $m-1$ st and *all* j . Let $c_m \neq 0$. Select γ_i such that $|\gamma_i| \leq 1/k^j$ and $|\gamma_i| \leq \mu^{m+2}$. There exists l such that $|\gamma_i| \leq 1/(k^{j+l})$.

$$\frac{1}{\gamma_i^m c_m} [\phi(\gamma_i x) - \phi(0)] = \frac{c_1}{c_m} x \gamma_i^{1-m} + \cdots + \frac{c_{m-1}}{c_m} x^{m-1} \gamma_i^{1-m} + x^m \\ + \frac{c_{m+1}}{c_m} x^{m+1} \gamma_i + \cdots.$$

The first $m-1$ terms of the right-hand side are either zero or can be approximated with error $O(k^{-j})$ by our induction hypothesis: for example we can approximate

$$\frac{c_1}{c_m} x \gamma_i^{1-m} \quad \text{by} \quad \frac{c_1}{c_m} \gamma_i^{1-m} \left[x + O\left(\frac{1}{(k^{m(j+l)})}\right) \right] = \frac{c_1}{c_m} \gamma_i^{1-m} x + O\left(\frac{1}{k^{l+j}}\right).$$

The $m+1$ st terms and after are

$$\gamma_i \left(\frac{c_{m+1}}{c_m} x^{m+1} \right) + \frac{\gamma_i}{c_m} \sum_{k=m+2}^j c_k x^k \gamma_i^{k-m-1}.$$

Now for $x \in [\alpha, \beta]$,

$$\sum_{k=m+2}^{\infty} |c_k x^k \gamma_i^{k-m-1}| \leq \sum_{k=m+2}^{\infty} |c_k x^k| |\gamma_i|^{k/(m+2)} \leq \sum_{k=m+2}^{\infty} |c_k| |\mu x|^k < \infty.$$

APPLICATIONS

1. If all coefficients c_k of the series are nonzero, the set of powers with nonzero coefficients is fundamental in $C[\alpha, \beta]$.

2. If $\phi(0) = 0$ and all but the zeroth coefficient of the series are nonzero, then the set of powers with nonzero coefficients is fundamental in $C_0[\alpha, \beta]$, the set of continuous functions on $[\alpha, \beta]$ vanishing at zero.

3. If ϕ is even and all even coefficients of the series are nonzero, the set of powers with nonzero coefficients is fundamental in $C[0, \beta]$.

4. If ϕ is odd and all odd coefficients of the series are nonzero, the set of powers with nonzero coefficients are fundamental in $C_0[0, \beta]$.

The first is a consequence of the Weierstrass theorem. The second can be deduced from the first. The third and fourth are consequences of the Müntz theorem.

GENERALIZATIONS

More generally we may be given ϕ continuous on an interval J containing zero and given a finite interval $[\alpha, \beta]$. A (ϕ, J) polynomial on $[\alpha, \beta]$ is any finite linear combination of terms of the form $a\phi(bx)$ such that $bx \in J$ for all $x \in [\alpha, \beta]$. Let $\{\gamma_k\} \rightarrow 0$. Then there exists K such that for $k \geq K$, $\gamma_k x \in J$ for all $x \in [\alpha, \beta]$. It is trivial to extend the theorem to cover this case.

Let $\phi(0) = 0$ and all but the zeroth coefficient of the Taylor series for ϕ be nonzero. Let $\{\gamma_k\}$ be a nonzero sequence with limit 0. Let f be given then $f - f(0) \in C_0[\alpha, \beta]$ and can be uniformly approximated by linear combinations of $\{\phi(\gamma_1 x), \phi(\gamma_2 x), \dots\}$. Hence the set of linear combinations of $\{1, \phi(\gamma_1 x), \phi(\gamma_2 x), \dots\}$ is dense in $C[\alpha, \beta]$.

Similar arguments show that if ϕ is odd and all odd coefficients of the Taylor series are nonzero, the set of linear combinations of $\{1, \phi(\gamma_1 x), \phi(\gamma_2 x), \dots\}$ is dense in $C[0, \beta]$.

Let ϕ be odd with the odd coefficients of its Taylor expansion nonzero. Let ψ be even with the even coefficients of its Taylor expansion nonzero. A linear combination of terms of the type $\phi(ax)$ or $\psi(bx)$ is called a (ϕ, ψ) -polynomial. The set of (ϕ, ψ) polynomials is dense in $C[\alpha, \beta]$.

REFERENCES

1. D. BRAESS, Chebyshev approximation by γ -polynomials, *J. Approximation Theory* **9** (1973), 20–43.
2. C. DE BOOR, On the approximation by γ -polynomials, in "Approximations with Special Emphasis on Spline Functions" (I.J. SCHOENBERG, Ed.), pp. 157–183, Academic Press, New York, 1969.
3. C. R. HOBBY AND J. R. RICE, Approximation from a curve of functions, *Arch. Rat. Mech. Anal.* **24** (1967), 91–106.